Deductive systems with unified multiple-conclusion rules

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Introduction

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(a) We prove statements asserting or rejecting a given proposition;
(b) We use the multiple-conclusion rules which premises and conclusions are finite sets of statements.

Introduction

It is due Łukasiewiecz that rejection was explicit including to logic. In the introduction to his paper[1] , he wrote:

"The concepts of "truth", "falsehood", and "assertion" I owe to Frege. In adding "rejection" to "assertion" I have followed Brentano."

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According to Brentano and in contrast to Frege, assertion (or acceptance) and rejection (or refutation, or denial) should have the same status. Let us note that assertion of a negation is much stronger than the rejection. For instance, in the Classical Logic we reject formula p (in symbols $\dashv p$), but the assertion $\vdash \neg p$ does not hold.

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modus tolens:	$\vdash (A \to B), \dashv B / \dashv A$	(MT)
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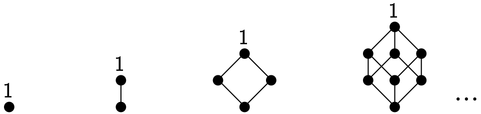
Carnap's motivation to introducing refutations and multiple-conclusion rules was requirement of categoricity: if we want to syntactically characterize two-valued classical semantics, this syntactical system should be valid only (up to matrix isomorphisms) on the two-element Boolean matrix. But any axiom and the rule which is valid in $(\mathbf{2}, \{1\})$, is valid in all matrices $(\mathbf{2}^n, \{1\}), n \geq 0$ as well.



$\mathbf{2}^0$ $\mathbf{2}$ $\mathbf{2}^2$ $\mathbf{2}^3$ ...

Introduction

Carnap's solution is to use refutations and multiple-conclusion (multiple-alternative) rules – the ordered pairs Γ/Δ of finite sets of formulas.

In semantics, a rule Γ/Δ is valid in matrix $(\mathbf{A}, D)$ if for any valuation ν,

$$\nu(\Gamma) \subseteq D \text{ entails } \nu(\Delta) \cap D \neq \varnothing.$$

A rejected (refuted) proposition $\dashv A$ is valid in a given matrix, if for some valuation, the value of A is not designated. For instance, $\dashv p$, where p is a variable, is valid in any matrix having at least one non-designated element, and $\dashv p$ is invalid in all matrices in which every element is designated.

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Before we proceed, the warning:

We do not consider multiple-conclusion logics in the sense of Shoesmith and Smiley or Carnap's junctives.

We use multiple-conclusion rules merely as means of derivation of a statement from a set of statements.

Outline

Unified Logic

We assume that Frm is a set of propositional formulas built in a regular way from a countable set *Var* of propositional variables and a finite set of connectives Ω.

Definition

A ***unified logic*** is an ordered pair $(\mathsf{L}^+, \mathsf{L}^-)$, where L^+ is a set of formulas closed under the rule of substitution: $\mathsf{Sb} := A/\sigma(A)$, where $A \in \mathsf{Frm}$ and σ is a substitution, while L^- is a set of formulas closed under the rule of reverse substitution: $\mathsf{Rs} := \sigma(A)/A$.

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For example, let Cl^+ be a set of all classical tautologies and $\mathsf{Cl}^- := \mathsf{Frm} \setminus \mathsf{Cl}^+$. Then the pair $\mathsf{UCL} := (\mathsf{Cl}^+, \mathsf{Cl}^-)$ is a unified classical logic.

Unified Logic

L^+ is a set of *asserted* (accepted) propositions – *theorems* ; L^- is a set of *rejected* (refuted, denied) propositions – *anti-theorems* ;

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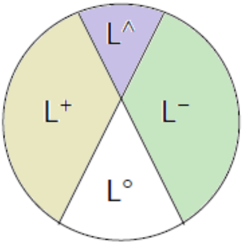
We make no assumptions regarding relations between L^+ and L^-. All possibilities are admissible:

L^+ - asserted propositions
L^- - rejected propositions
$L^\wedge$ - asserted and rejected propositions
L° - neither asserted nor rejected propositions



Introduction: Types of unified logics

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A full and coherent logic is called *standard.*

Example. Let us take the three-element Heyting algebra $\mathbf{A} := (\{\mathbf{0}, \mathsf{a}, \mathbf{1}\}; \to, \wedge, \vee, \neg)$, and consider a (logical) matrix $\mathcal{M} := (\mathbf{A}; D^+ = \{\mathbf{1}\}, D^- = \{\mathbf{0}\})$. For any proposition A, we let

$$A \in \mathsf{L}^+ \iff \text{for each valuation } \nu, \nu(A) \in D^+;$$
$$A \in \mathsf{L}^- \iff \text{there is a valuation } \nu, \text{ such that } \nu(A) \in D^-.$$

Then, $A \in \mathsf{L}^+$ if and only if A is valid in the Smetanich logic. $A \in \mathsf{L}^-$ if and only is A is invalid in the Classical logic. Propositions $p \vee \neg p$ and $\neg\neg p \to p$ are neither asserted, not rejected.

Unified Logic

It is custom to define logic by a consequence relation. If assertions and rejections have the same status, we need to consider the consequence relations on sets of meta-statements of the type "A is asserted" ($A \in \mathsf{L}^+$) and "A is rejected" ($A \in \mathsf{L}^-$).

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It is inconvenient for our purposes to use $\vdash$ and $\dashv$ for "is asserted" and "is rejected", because the notation like

$$\vdash A_1, \ldots, \vdash A_n \vdash \vdash B$$

looks confusing. Instead, we use $\oplus A$ for "A is asserted", and $\ominus A$ for "A is rejected". The notation[3]

$$\oplus A_1, \ldots, \oplus A_n \vdash \oplus B$$

is less confusing.

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Unified consequence relation is a binary relation $\vdash$ defined on sets of statements and statements and satisfying the regular properties of consequence relation: for any sets $\Gamma, \Delta \subseteq \mathcal{S}$ and any $\alpha, \beta \in \mathcal{S}$

$$\alpha \vdash \alpha \qquad (R)$$
$$\text{if } \Gamma \vdash \alpha \text{ and } \Gamma \subseteq \Delta, \text{ then } \Delta \vdash \alpha \qquad (M)$$
$$\text{if } \Gamma \vdash \alpha \text{ and } \alpha, \Delta \vdash \beta, \text{ then, } \Gamma, \Delta \vdash \beta. \qquad (T)$$

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Relation $\vdash$ is *finitary* if $\Gamma \vdash \alpha$ entails $\Gamma' \vdash \alpha$ for some finite $\Gamma' \subseteq \Gamma$.

Logic: theorems

Each unified consequence relation $\vdash$ defines a set of *asserting theorems* :

$$\mathsf{Th}^{+}(\vdash) := \{\alpha \in \mathcal{S}^{+} \mid \ \vdash \alpha\}$$

and a set of *refuting theorems* :

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But if we consider the projections onto the set of propositions:

$$\mathrm{L}^{+} := \{A \in \mathrm{Frm} \mid \ \oplus A \in \mathrm{Th}^{+}(\vdash)\},$$
$$\mathrm{L}^{-} := \{A \in \mathrm{Frm} \mid \ \ominus A \in \mathrm{Th}^{-}(\vdash)\}$$

the situation is different.

Introduction: types of refutation

In general, there are two ways of how to handle refutation syntactically: direct and indirect. To determine whether formula A is refutable one can do one of two things:

(a) to derive in a meta-logic a statement about refutability of A (Ł-proof - Łukasiewicz-style proof)

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An existence of an Ł-proof entails the existence of i-proof. The converse is true under some assumptions (some weak form of the deduction theorem[4]) and we will revisit this issue later.

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Introduction: Direct refutation, an example

As an example, let us consider a Classical Propositional Calculus (CPC) with regular set of axioms and rules

$$\oplus A, \vdash \oplus(A \to B) / \oplus B \qquad \text{(MP)}$$
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And let us extend this calculus to calculus CPC° by adding an anti-axiom

$$\vdash \ominus p,$$

where p is a propositional variable, and two rules

$$\oplus(A \to B), \ominus B/\ominus A \qquad \text{(MT)}$$
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Soundness easily follows from the observation that all axioms, the anti-axiom and the rules are valid in the 2-element Boolean algebra.

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We can use the semantic means and conclude that

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If $I \nvdash A$ and $I \nvdash \neg\neg A$, then $I^\circ \nvdash \oplus A$ and $I^\circ \nvdash \ominus A$. Thus,

$$\mathsf{L}^+(I^\circ) \cup \mathsf{L}^-(I^\circ) \neq \mathsf{Frm},$$

that is, $\mathfrak{I}$ is not full.

Outline

Multiple-Alternative Rules

If Γ, Δ are finite sets of meta-statements, an ordered pair Γ/Δ is called a *structural multiple-conclusion* or *multiple-alternative* rule (m-rule for short). The premises Γ are viewed conjunctively, while the conclusions Δ are viewed disjunctively.

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In general, we divide rules into three categories: if $r := \Gamma/\Delta$ is a rule, then

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For instance, $\oplus p, \oplus(p \to q)/ \oplus q$ is a conclusive rule; $\oplus(p \vee q)/ \oplus p, \oplus q$ is an inconclusive rule; $\oplus p, \ominus p/\varnothing$ is a terminating rule.
In addition to m-rules, we consider two rules: the rule of substitution Sb, and the rule of reverse substitution Rs.

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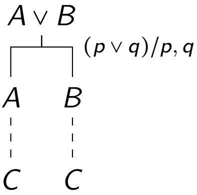

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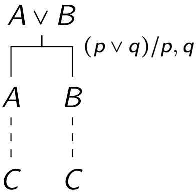

By applying rule Γ/Δ we get the alternatives Δ to be considered separately.

Multiple-Alternative Inference

We use $\blacktriangledown$ to denote an empty set of premises, and $\blacktriangle$ to denote an empty set of alternatives. $\blacktriangledown$ and $\blacktriangle$ are merely notations and they are not the symbols of the language or meta-language.

[5] Note that we define an inference from (Γ, R) without clarifying what we are deriving.

Multiple-Alternative Inference

We use ▾ to denote an empty set of premises, and ▴ to denote an empty set of alternatives. ▾ and ▴ are merely notations and they are not the symbols of the language or meta-language.

Inferences are finite trees the nodes of which are labeled by statements, ▾ or ▴. A leaf labeled by ▴ is *teriminating* (we have reduced a case to contradiction), otherwise, it is *extendable* .

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Let R be a set of rules (that may include Sb and/or Rs) and Γ be a set of statements (which may be empty). An *inference from* Γ *by* R (or (Γ, R)-*inference* for short) is a finite tree nodes of which are labeled by statements, and it is defined by induction[5]:

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Like in a Hilbert-style inference, we use the assumptions and apply the inference rules.

A tree consisting of a single node (a root) labeled by $\blacktriangledown$ is a (Γ, R)-inference (*it is needed for a sake of convenience*).

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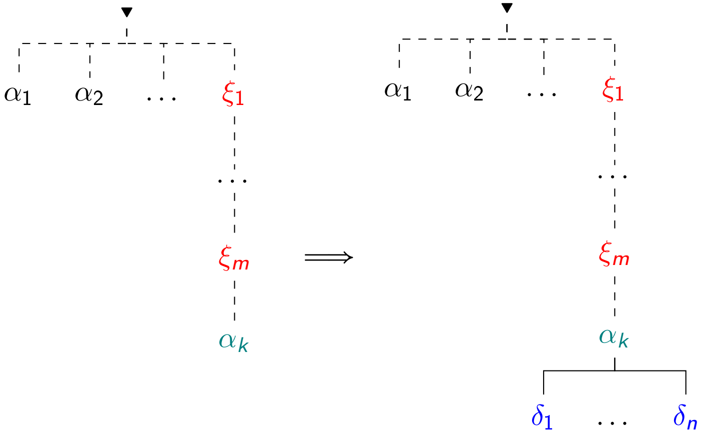
Applying the rules: if $\mathfrak{I}$ is a (Γ, R)-inference, then any non-terminal leaf λ can be extended by adjoining the leaves labeled by $\blacktriangle$, or by statements from a finite set Δ, provided there is an instance $\Xi/\blacktriangle$ or Ξ/Δ of a rule from R, and all statements from Ξ are between λ and the root. The tree obtained in such a way is a (Γ, R)-inference.

Multiple-Alternative Inference

Suppose that $\frac{\xi_1,\ldots,\xi_m}{\delta_1,\ldots,\delta_n}$ is an instance of a rule from R.



Multiple-Alternative Inference

Let Γ, Δ be sets of statements, α be a statement and R be a set of rules.

Definition

α *is derivable from* Δ *by* (Γ, R), if there is a $(\Delta \cup \Gamma, \mathsf{R})$-inference each leaf of which is labeled by α or by $\blacktriangle$.

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Roughly speaking, α is derivable from Δ if after we have considered every case arisen in the proof, we either have derived α, or we have arrived at a contradiction, meaning, that the case is not possible.

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Proposition

Any pair consisting of a set of statements Γ *and a set of rules* R*, defines a consequence relation:*

$$\Delta \vdash \alpha \leftrightharpoons \alpha \textit{ is derivable from } \Delta \textit{ by } (\Gamma, \mathrm{R}).$$

Multiple-Alternative vs. Classical Inference: an Example

Сова приложила ухо к груди Буратино.
- Пациент скорее мертв, чем жив, - прошептала она.

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- Касторкой! - презрительно захохотала Сова.
- Или касторкой, или не касторкой, - проскрежетал Богомол.

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Definition

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Let $\mathsf{L} = (\mathsf{L}^+, \mathsf{L}^-)$ be a unified logic. A deductive system $\mathcal{D}$ is *Ł-complete* for L, or L is *defined by* $\mathcal{D}$, if

$$\vdash_{\mathcal{D}} \alpha \iff \begin{cases} \alpha = \oplus A \text{ and } A \in \mathsf{L}^+ \\ \alpha = \ominus A \text{ and } A \in \mathsf{L}^-. \end{cases}$$

Any unified deductive system defines a unified logic.

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Any unified deductive system defines a unified logic.

If $\mathcal{D}$ contains only positive rules, it is *C-complete* for L, if

$$\vdash_{\mathcal{D}} \alpha \iff \begin{cases} \alpha = \oplus A \text{ and } A \in L^+ \\ \alpha = \ominus A \text{ and } \oplus A \vdash_{\mathcal{D}} \ominus B, \text{ where } \ominus B \text{ is an anti-axiom.} \end{cases}$$

Admissible multiple-alternative rules

An m-rule Γ/Δ is *admissible* for a given unified logic L, if for each substitution that makes valid all statements from Γ, at least one statement from Δ is valid. $\blacktriangledown$ is considered being always valid, and $\blacktriangle$ is considered being always invalid.

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Proposition

In any intermediate logic, for any formula A,

$$\text{rule } \oplus A/\blacktriangle \text{ is admissible if and only if rule } \blacktriangledown/\oplus\neg A \text{ is admissible.}$$

The proof of $\Longleftarrow$ is trivial, while $\Longrightarrow$ follows immediately from the Glivenko Theorem.

Admissible multiple-alternative rules

In terms of admissible rules, we have the following:

(coherency) a logic is coherent if and only if the rule

$$\mathsf{Co} := \frac{\oplus p, \ominus p}{\blacktriangle} \text{ is admissible;}$$

(fullness) a logic is full if and only if the rule

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In what follows, the above m-rules play the central role.

For convenience, we use the notation:

$$\overline{\alpha} = \begin{cases} \ominus A, & \text{when } \alpha = \oplus A \\ \oplus A, & \text{when } \alpha = \ominus A. \end{cases}$$

Admissible multiple-alternative rules

Let L be a standard logic. Then, the following holds: for any finite sets Γ, Δ and any statement α,

if the rule $\dfrac{\alpha, \Gamma}{\Delta}$ is admissible, then the rule $\dfrac{\Gamma}{\overline{\alpha}, \Delta}$ is admissible;

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In other words, one can move a statement from premises to alternatives, or vice-versa, with changing the "sign" of the statement. For logics without rejection the above makes no sense.

Admissible multiple-alternative rules

Let L be a standard logic signature of which contains $\to$. If Modus Ponens is admissible for L, then, all the following eight variations of Modus Ponens are admissible:

$$\frac{\blacktriangledown}{\ominus p, \ominus(p \to q), \oplus q}; \quad \frac{\oplus p}{\ominus(p \to q), \oplus q}; \quad \frac{\oplus(p \to q)}{\ominus p, \oplus q}; \quad \frac{\ominus q}{\ominus p, \ominus(p \to q)};$$

$$\frac{\oplus p, \oplus(p \to q)}{\oplus q}; \quad \frac{\oplus p, \ominus q}{\ominus(p \to q)}; \quad \frac{\oplus(p \to q), \ominus q}{\ominus p}; \quad \frac{\oplus p, \oplus(p \to q), \ominus q}{\blacktriangle}.$$

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$$\frac{\oplus p, \oplus(p \to q)}{\oplus q}; \quad \frac{\oplus p, \ominus q}{\ominus(p \to q)}; \quad \frac{\oplus(p \to q), \ominus q}{\ominus p}; \quad \frac{\oplus p, \oplus(p \to q), \ominus q}{\blacktriangle}.$$

By the same argument, for the rule of substitution we have two variations that are either simultaneously admissible, or simultaneously not admissible:

$$\frac{\oplus A}{\oplus \sigma(A)}; \quad \frac{\ominus \sigma(A)}{\ominus A}$$

Derivations of rules

Let R be a set of rules and $r := \Gamma/\Delta$ be a rule. We say that r *is derivable* from R (in symbols $R \vdash r$), if there is a (Γ, R)-inference all leaves of which do not contain statements not from Δ.

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$$\mathsf{R}, \mathsf{r}' \vdash \mathsf{r}.$$

$\mathsf{r}' \vdash_{\mathsf{R}} \mathsf{r}$ means that in any inference, every application of rule r can be replaced with the suitable applications of rules R and r'. In other words, rule r can be eliminated from any inference and replaced by rules R, r'.

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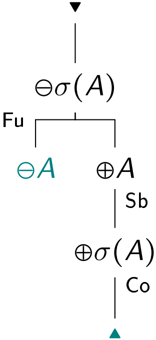
The rules Co and Fu allows to derive the different variations of the given rules from each other. Let

$$\mathcal{S} := \{\mathrm{Co}, \mathrm{Fu}\}.$$

Reduction of Rs to Sb

Proposition. Sb $\vdash_{\mathcal{S}}$ Rs.



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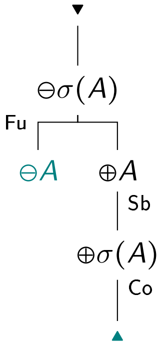
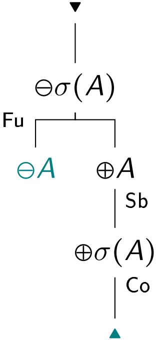


Thus, in each deductive system that has postulated rules Co, Fu and Sb, the rule Rs can be eliminated.

Reduction of Rs to Sb

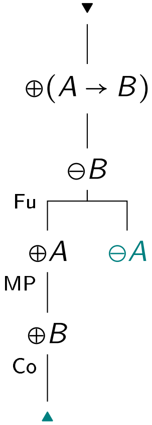
Proposition. $\mathsf{Sb} \vdash_{\mathcal{S}} \mathsf{Rs}$.



Thus, in each deductive system that has postulated rules Co, Fu and Sb, the rule Rs can be eliminated.

Ł-complete systems

Proposition. MP $\vdash_{\mathcal{S}}$ MT.



Ł-complete systems

Theorem

Let $\mathcal{D}$ be a deductive system containing only positive rules and the rule of substitution. Then, if $\mathcal{D}$ is C-complete for a unified logic L*, the system $\mathcal{D}'$ obtained from $\mathcal{D}$ by postulating* Co *and* Fu*, is Ł-complete.*

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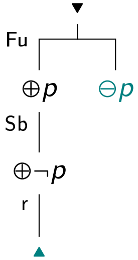
Example

One can take any calculus that defines the classical logic and contains the rule of substitution, and convert it to a C-complete deductive system by adding anti-axiom $\ominus p$. If we add to this deductive system Co and Fu, we obtain an Ł-complete system.

Ł-complete systems

Moreover, if we take any calculus with the rule of substitution defining the classical logic, we can convert it into an Ł-complete deductive system by adding the rules Co, Fu and r := $\oplus p, \oplus\neg p/\blacktriangle$. The needed anti-axiom $\ominus p$ is derivable:



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Corollary

Let $(\Gamma, \mathsf{R} \cup \mathcal{S})$ *be a deductive system defining a unified logic* L*. Then there is a system of positive rules* R^{+}*, such that* $(\Gamma, \mathsf{R}^{+} \cup \mathcal{S})$ *defines* L*.*

Ł-complete system for the Classical Logic

Theorem

The deductive system consisting of the below rules[a] is Ł-complete for the classical logic Cl.

(i)	$\mathsf{Ei} = \dfrac{\oplus p,\ \oplus(p\to q)}{\oplus q}$	$\mathsf{Ii}_1 = \dfrac{\oplus q}{\oplus(p\to q)}$	$\mathsf{Ii}_2 = \dfrac{\oplus(p\to(q\to r))}{\ominus(p\to q),\ \oplus(p\to r)}$
(c)	$\mathsf{Ecl} = \dfrac{\oplus p\wedge\oplus q}{\oplus p}$	$\mathsf{Ecr} = \dfrac{\oplus p\wedge\oplus q}{\oplus q}$	$\mathsf{Ic} = \dfrac{\oplus p, \oplus q}{\oplus(p\wedge q)}$
(d)	$\mathsf{Edl} = \dfrac{\ominus(p\vee q)}{\ominus p}$	$\mathsf{Edr} = \dfrac{\ominus(p\vee q)}{\ominus q}$	$\mathsf{Id} = \dfrac{\oplus(p\to r), \oplus(q\to r)}{\oplus((p\vee q)\to r)}$
(n)	$\mathsf{En} = \dfrac{\oplus p,\ \oplus\neg p}{\blacktriangle}$	$\mathsf{In} = \dfrac{\blacktriangledown}{\oplus p,\ \oplus\neg p}$	
(r)	$\mathsf{Co} = \dfrac{\oplus p,\ \ominus p}{\blacktriangle}$	$\mathsf{Fu} = \dfrac{\blacktriangledown}{\oplus p,\ \ominus p}$	$\mathsf{Sb} = \dfrac{\oplus A}{\oplus\sigma(A)}$

[a] The positive m-rules that define the positive part of Cl are are similar to m-rules from Shoesmith and Smiley, Multiple-conclusion logic,2008.

Final remarks

The rule $\blacktriangledown / \oplus p, \ominus p$ (and not the $\blacktriangledown / \oplus p, \oplus \neg p$, or $\blacktriangledown / \oplus (p \vee \neg p)$) expresses the Law of Excluded Middle. The Law of Excluded Middle is not about disjunction and negation: you may have it for the systems without disjunction and negation. The Law of Excluded Middle means that

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Accordingly, the rule $\oplus p, \ominus p / \blacktriangle$ expresses the Law of Non-Contradiction, which is not about conjunction and negation; it means that

One cannot assert and reject the same proposition at the same time.

Thanks

Thank you for your patience and attention.

Alex Citkin