Proto-Entailment in RS logic

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A jump into abstraction – performed in universal algebra and universal logic – allows space for monsters.
J-Y. Bézeau

ABSTRACT. In this paper I propose a formalization of proto-entailment relation introduced by V. Shalak by means of RS logic. The first section clarifies the idea and formal developments of RS logic, which is the logic of Rational Subject. In the second section I will very briefly introduce the conception of proto-entailment as it was promoted in Shalak's writings. The third section contains the formal account for proto-entailment and axiotimatization of resulting logic.

Keywords: proto-entailment, logic of rational subject, generalized truth-values

1 Logic of Rational Subject

The abbreviation *RS-logic* expansion is [logic] 'of Rational Subject', that is a four-valued propositional logic, whose values are two-component entities composed of logical and epistemological constituents. First the idea of such a logic emerged in the course of working on the project of generalized classical truth values [4]. We elaborated an idea of distinguishing between ontological and epistemological aspects of classical truth values. In so doing, we came across two unary twin connectives that deal only with either ontological or epistemological component of generalized classical truth value, leaving the other untouched. That is why these connectives were labeled as semi-classical negations. I turned onto whether there

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is any logic wherein any of our semi-negations is treated as full-scale one. This stream of thought led me to the logic of Rational Subject.

Imagine a rational subject who knows the laws of classical logic. It means that when it is necessary to calculate the value of any compound formula from the values of its constituent formulas our subject performs computations guided by a knowledge of classical truth-assignments. It is evident that proceeding along these lines he (or she) sometimes can figure out the value of a formula, and, thus, *knows* its value, and sometimes the information in hand is not enough to fix the value of the formula, and when this occurs, our subject *does not know* the value. Hence, besides two 'logical' (or ontological) values Truth and Falsity one must take into account two extra 'epistemic' values characterizing the state of rational subject's knowledge. Let Truth and Falsity as usual be denoted by 't' and 'f', while for 'knows' and 'does not know' we select '1' and '0' correspondingly. Then we have just four values being two-component entities composed of logical and epistemological constituents that can be treated as pairs or as sets:

T1 $\langle t, 1\rangle$ $\{t, 1\}$;
T0 $\langle t, 0\rangle$ $\{t\}$;
F0 $\langle f, 0\rangle$ $\{f\}$;
F1 $\langle f, 1\rangle$ $\{f, 1\}$.

Consider the clauses for negation and conjunction to clarify the way rational subject works. If rational subject knows that an arbitrary formula is true, he knows that its negation is false, and vice versa. In the meantime, if you do not know the value of a formula, you do not know the value of its negation. The resulting truth-table goes as follows in Figure 1.

A	$\neg A$
T1	**F1**
T0	**F0**
F1	**T1**
F0	**T0**

Figure 1. Table for 'rational' semi-classical negation

Due to its classical nature conjunction is true if and only if both conjuncted formulas are true, that determines the first component of values-as-pairs (occurrence of element 't' in a value-as-set). The second epistemic component of a value can be calculated on the basis of the following reflections: one knows that conjunction is true if and only if one knows that both conjuncts are true, and one knows that conjunction is false if and only if one knows that at least one of conjuncts is false. Summing up these considerations we receive the truth-table for conjunction depicted in Figure 2. The analogous

$\wedge$	**T1**	**T0**	**F0**	**F1**
T1	**T1**	**T0**	**F0**	**F1**
T0	**T0**	**T0**	**F0**	**F1**
F0	**F0**	**F0**	**F0**	**F1**
F1	**F1**	**F1**	**F1**	**F1**

Figure 2. Table for 'rational' conjunction

argument provided for disjunction clause makes it possible to consider the structure of generalized values as a four-elements lattice with linear ordering depicted in Figure 3.

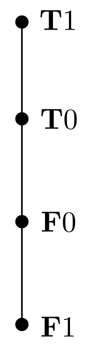

Figure 3. The 'rational' lattice

Now define a valuation function v as a map from the set of propositional variables to the set $V = \{T1, T0, F0, F1\}$, and in a straight forward way extend it to arbitrary formula A given correspondence between lattice meet (join) and conjunction (disjunction). Thus, we have a four-valued valuational system, which allows to define different consequence relations on it.

For a period I had been zeroing in on the different problems put RS logic aside. However it was my student Yekaterina Kubyshkina who in her graduation thesis and relevant publications [3] examined some consequence systems, which axiomatize RS logics with different consequence relations. In particular, she introduced three consequence relations: $\forall A, B$

- $A \models_{RM} B \Leftrightarrow \forall v(v(A) \in D \Rightarrow v(A) \in D)$,
  where $D = \{T1\}$ — T1-preserving consequence;
- $A \models_{TV} B \Leftrightarrow \forall v(v(A) \in D \Rightarrow v(A) \in D)$,
  where $D = \{T1, T0\}$ — truth-preserving consequence;
- $A \models_{KL} B \Leftrightarrow \forall v(v(A) \leq v(A))$,
  where $\leq$ is the 'rational' order — comparative consequence.

She has proved that corresponding semantical logics can be presented as consequence systems: RS_{RM} (RS with $\models_{RM}$) is axiomatized by the first-degree fragment of RM; RS_{TV} (RS with $\models_{TV}$) is axiomatized by classical consequence system; RS_{KL} (RS with $\models_{KL}$) is axiomatized by Kleene strong logic.

Immediately, a string of questions arises. And among them the following directly pertains to the topic of this paper: what is the consequence system axiomatizing a 1-preserving entailment? To answer this question we first turn to Shalak's idea of proto-entailment.

2 Proto-Entailment

Modern Russian logician Vladimir Shalak set forth an idea of proto-entailment proceeding from radically different intuitive premisses. The title of his doctoral dissertation is 'Proto-Logic: new insight into the nature of logicality' and he sees his primary objective in clarifying the very concept of logic. His approach is very close to the so called project of universal logic, which pretends to be a general theory of logics. Shalak himself highlights the cognation of his theory with the ideas of J-Y. Bésiau [1, 2]. J-Y. Bésiau interprets universal logic by analogy with universal algebra: the latter is an abstract set of formulas together with equally abstract consequence relation subject for no specific restrictions.

Meanwhile, it is worth noting that Shalak himself in English-language abstract to his papers (published in Russian) uses the term 'consequence relation'. However he oftentimes emphasizes that this consequence relation is free from well-known paradoxes, and hence makes a good name of (proto-) entailment for it.

However even such an abstract relation needs a precise definition. This is a fragment from his paper devoted to an alternative definition of consequence relation that helps to grasp the underlying informal intuition.

> In classical logic, truth of premises is a sufficient condition of verity of the conclusion. However, that is too stronghold limiting a requirement. A laxer claim might be to have valid ways of reasoning simply not lead us to erroneous conclusion or fallacies...
>
> In other words, the form of the argument is valid if the knowledge of its premises' truth-value is a sufficient condition of the awareness of its conclusion's truth-value... [5, p. 283].

To understand why and how Shalak proceeded from this informal motivation to the axiomatically presented proto-Boolean logic, one should take into account the other, maybe even most important for him, idea that constitutes his conception of modern logic. One of his fundamental presumptions is that the radical turn from subject-predicate paradigm in logic to relational one neither was necessarily determined nor offered any advantage in a formal language expressive power. In fact, he suspects that as a result of such a paradigm shift our world-view has been distorted. It would be more natural and convenient to develop symbolic logic on the ontological basis of (monadic) properties and functions rather than on the ground of relational structures. This presuppositions have strongly influenced on his further formal explication of consequence relation, which he defines functionally as follows.

> This [the above consideration] gives rise to the following definition of the entailment relation:

> The set of formulas $\Sigma = \{B_1, \ldots, B_k\}$ entails formula A, iff there exists function f which allows calculation of the truth-value of A given truth-values of the formulas of the set Σ. [5, p. 283].

Quite predictably this function f turns to be Boolean one that provides extremely plain axiomatization of proto-Boolean logic as a consequence system **ACL** (that is *alternative consequence logic* or *alternative to classical logic* , as may well be imagined). There are just three axiom schemes and two rules:

A1 $A \vee \neg A$;

A2 $\{A, B\} \vdash_{SH} A \wedge B$;

A3 $\{A\} \vdash_{SH} \neg A$;

R1 $\dfrac{\vdash_{TV} A \equiv B}{\{A\} \vdash_{SH} B}$; **R2** $\dfrac{\Gamma \vdash_{TV} A,\ \{A\} \cup \Delta \vdash_{SH} B}{\Gamma \cup \Delta \vdash_{SH} B}$,

where $\vdash_{SH}$ and $\vdash_{TV}$ stand for Shalak's proto-entailment and classical consequence relation correspondingly.

It can be easily shown that $\vdash_{TV} A \Rightarrow \vdash_{SH} A$, thereby validating in **ACL** all classically valid formulas.

In his resent writings, Shalak makes an attempt to formalize more abstract *functional* concept of proto-entailment. However, currently he has only suggestive axiomatization of corresponding consequence relation.

3 Proto-Entailment as a 1-Preserving Consequence Relation

In what follows, I will present a consequence system RS_{PE} which formalizes proto-entailment as a 1-preserving consequence relation in RS logic. In so doing, first consider semantics for RS_{PE} in more detail.

For the sake of convenience, in this section, the values of RS logic will be interpreted as sets (sf. the 1st section). A valuation function v is the map from the set of propositional variable into the four-element set of values-as-pairs, extended to compound formulas in a straightforward way as provided by the truth-tables above. Then the following proposition can be put forward.

Proposition 1.
$t \in v(\neg A) \Leftrightarrow f \in v(A) \quad f \in v(\neg A) \Leftrightarrow t \in v(A)$
$1 \in v(\neg A) \Leftrightarrow 1 \in v(A)$
$t \in v(A \wedge B) \Leftrightarrow t \in v(A)$ *and* $t \in v(A)$
$f \in v(A \wedge B) \Leftrightarrow f \in v(A)$ *or* $f \in v(B)$
$1 \in v(A \wedge B) \Leftrightarrow [1 \in v(A)$ *and* $1 \in v(B)$ *and* $t \in v(A)$ *and* $t \in v(B)]$ *or* $[1 \in v(A)$ *and* $f \in v(A)]$ *or* $[1 \in v(B)$ *and* $f \in v(B)]$.

Definition 1. For arbitrary formulas A and B of L_{RS}, $A \models_1 B \Leftrightarrow \forall v(1 \in v(A) \Rightarrow 1 \in v(B))$.

A consequence system RS_{PE} is presented as pair $(L_{RS}, \vdash)$, where $\vdash$ satisfies the following deductive postulates:

A1. $A \vdash \neg A$

A2. $\neg A \vdash A$

A3. $A \wedge (B \vee C) \vdash (A \wedge B) \vee C$

A4. $A \wedge B \vdash \neg A \vee \neg B$

A5. $\neg A \vee \neg B \vdash A \wedge B$

A6. $A \vee B \vdash \neg A \wedge \neg B$

A7. $\neg A \wedge \neg B \vdash A \vee B$

R1. $\dfrac{A \vdash B, B \vdash C}{A \vdash C}$ **R2.** $\dfrac{A \vdash B, A \vdash C}{A \vdash B \wedge C}$

R3. $\dfrac{A \vdash B, C \vdash B}{A \vee C \vdash B}$ **R4.** $\dfrac{A \vdash B, A \vdash_{TV} \neg B}{A \vdash B \wedge C}$

R5. $\dfrac{A \vdash B \wedge C, A \vdash_{TV} B \wedge C}{A \vdash B, A \vdash C}$ **R6.** $\dfrac{A \vdash B \wedge C, A \vdash_{TV} \neg B}{A \vdash B \text{ or } A \vdash C}$,

where $\vdash_{TV}$ designates classical consequence relation.

There are some interesting and helpful theorems:

t1. $A \wedge B \dashv\vdash \neg(\neg A \vee \neg B)$

t2. $A \vee B \dashv\vdash \neg(\neg A \wedge \neg B)$

t3. $A \wedge \neg A \dashv\vdash A$

t4. $C \wedge (A \wedge \neg A) \dashv\vdash C \wedge A$

t5. $A \vdash A$

The proof of soundness is mostly a routine check which can be smoothly omitted.

THEOREM 1 (COMPLETENESS). *For any A and B of L_{RS}: If $A \models_1 B \Leftrightarrow A \vdash B$.*

PROOF. Suppose $A \models_1 B$. To show that $A \vdash B$, define canonical valuation via consequence relation as follows:
$v_c(p) = T1 \Leftrightarrow A \vdash_{TV} p$ *and* $A \vdash p$;
$v_c(p) = T0 \Leftrightarrow A \vdash_{TV} p$ *and* $A \nvdash p$;
$v_c(p) = F0 \Leftrightarrow A \vdash_{TV} \neg p$ *and* $A \nvdash p$;
$v_c(p) = F1 \Leftrightarrow A \vdash_{TV} \neg p$ *and* $A \vdash p$.
To simplify the proof consider only three generalized conditions for canonical valuation so defined:

1. $t \in v_c(p) \Leftrightarrow A \vdash_{TV} p$;
2. $f \in v_c(p) \Leftrightarrow A \vdash_{TV} \neg p$;
3. $1 \in v_c(p) \Leftrightarrow A \vdash p$.

Now we need to prove that the canonical valuation for arbitrary formula B satisfies conditions 1–3.

The first point that strikes the eye is the possibility for conditions 1 and 2 to coincide. If $A \vdash_{TV} B$ and $A \vdash_{TV} \neg B$ then $A \vdash_{TV} B \wedge \neg B$. The latter means that A is of the form $F \wedge (C \wedge \neg C)$. By **t3.** and **t4.**, $F \wedge (C \wedge \neg C) \dashv\vdash F \wedge C$. Let now A^* be $F \wedge C$. As long as according to our basic assumption $A \models_1 B$, to show that $A \vdash B$ is equivalent to show that $A^* \vdash B$, reducing the case to non-contradictory one.

The proof for arbitrary formula B will be carried out by simultaneous induction on the length of a formula. Keeping in mind theorems **t1.** and **t2.** one can consider only cases with negation and conjunction.

Case $(\neg B)$ and conditions 1–3 hold.

1) $t \in v_c(\neg B) \Leftrightarrow_{[prop.1.]} f \in v_c(B) \Leftrightarrow_{[ind.\ assumption]} A \vdash_{TV} \neg B$

2) $f \in v_c(\neg B) \Leftrightarrow_{[prop.1.]} t \in v_c(B) \Leftrightarrow_{[ind.\ assumption]} A \vdash_{TV} B$
$\Leftrightarrow_{[PC]} A \vdash_{TV} \neg\neg B$

3) $1 \in v_c(\neg B) \Leftrightarrow_{[prop.1.]} 1 \in v_c(B) \Leftrightarrow_{[ind.\ assumption]} A \vdash B$
$\Leftrightarrow_{[A1.,A2]} A \vdash \neg B$

Case $(B \wedge C)$ and conditions 1–3 hold.

1) and 2) are routine.

3) $1 \in v_c(B \wedge C) \Leftrightarrow_{[prop.1.]} [1 \in v_c(B)\ and\ 1 \in v_c(C)\ and$
$t \in v_c(B)\ and\ t \in v_c(C)]\ \ or\ [1 \in v_c(B)\ and\ f \in v_c(B)]$
$or [1 \in v_c(C)\ and\ f \in v_c(C)]$

$\Rightarrow$:

$1 \in v_c(B)\ and\ 1 \in v_c(C)\ and\ t \in v_c(B)\ and\ t \in v_c(C) \Rightarrow_{[ind.\ assum.]}$
$A \vdash B\ and\ A \vdash C \Rightarrow_{[R.2.]} A \vdash B \wedge C$

$1 \in v_c(B)\ and\ f \in v_c(B) \Rightarrow_{[ind.\ assum.]} A \vdash B\ and\ A \vdash_{TV} C \Rightarrow_{[R.4.]}$
$A \vdash B \wedge C$

$1 \in v_c(B)\ and\ f \in v_c(B)$ is analogous

$\Leftarrow$: $A \vdash B \wedge C \Rightarrow_{[R.5.]} (A \vdash B, A \vdash C, A \vdash B \wedge C)\ or$
$(A \vdash B, A \vdash_{TV} \neg B)\ or\ (A \vdash C, A \vdash_{TV} \neg C) \Rightarrow_{[ind.\ assum.]}$
$[1 \in v_c(B)\ and\ 1 \in v_c(C)\ and\ t \in v_c(B)\ and\ t \in v_c(C)]$
$or\ [1 \in v_c(B)\ and\ f \in v_c(B)]\ or\ [1 \in v_c(C)\ and\ f \in v_c(C)] \Leftrightarrow_{[prop.1.]}$
$1 \in v_c(B \wedge C)$

Turning to completeness, $A \models_1 B$ means that $\forall v(1 \in v(A) \Rightarrow 1 \in v(B))$. And hence $1 \in v_c(A) \Rightarrow 1 \in v_c(B)$, that is $A \vdash A \Rightarrow A \vdash B$. $A \vdash A$ holds by **t5.**, and by MP, $A \vdash B$, completing the proof. □

4 Conclusion

The aim of this paper was twofold.

(1) *Formalization of proto-entailment.* What kind of (proto-)entailment, in Shalak's sense, is formalized by the system RS_{PE}? Is it Boolean proto-entailment or general functional one? The answer that suggests itself is that the 1-preserving consequence relation corresponds to Boolean proto-entailment. The main argument in favour of such a conclusion is the apparently classical justification

of the truth-conditions for compound formulas exploited in the first section. Though one can call in question this reflection by asking if any other (non-classical) interpretation of propositional connectives are entitled to exist. Maybe it would be useful to consider non-Fregean logical components of compound values? As I see it, presumably this trick won't come off. The proliferation of logical components will complicate the assignment procedure and bring about to loss of clarity in compound values interpretation. Thus, the presented formalization can pretend to be the explication of general functional proto-entailment as well.

What is worth noting is the relationship between the 1-preserving entailment and the t-preserving entailment ($\models_{TV}$): if $A \models_1 B$, then $A \models_{TV} B$ or $A \models_{TV} \neg B$ but not vice versa. Supporting my previous claim the same relation holds between $\models_1$ and $\models_{RM}$ and between $\models_1$ and $\models_{KL}$!

(2) *Prospects for RS logic.* Digressing from proto-entailment, this logic seems of certain interest in itself.

First, it opens possibilities for a wide range of different consequence relations defined for instance via transfer from truth of premises to the awareness of conclusion and so on.

Second, *RS* logic is still waiting to be supplied with appropriate implication(s). Interestingly, natural classical style implication can be easily added as an abbreviation for $\neg A \vee B$. However this simple-mind implication turns to be a Kleene's one. Regarding Łukasiewicz's implication, which, if added, would allow to get the full-fledged Ł4, it does not agree with our informal intuition about the values of *RS*. For instance, consider the case when $\mathbf{2/3}$ was assigned to the antecedent, while the consequent has the value $\mathbf{1/3}$. In terms of *RS* logic values this assignment means **T0** for $\mathbf{2/3}$ and **F0** for $\mathbf{1/3}$. Under these circumstances the value of Łukasiewicz's implication will be $\mathbf{2/3}$, that looks at least strange.

Third, this logic is in a sense an epistemic one. It can be applied as a logical tool for public announcement modelling and other exciting ventures.

References

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ADDITION