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NEW LIGHT ON THE SQUARE OF OPPOSITIONS AND ITS NAMELESS CORNER

Abstract. *It has been pointed out that there is no primitive name in natural and formal languages for one corner of the famous square of oppositions. We have all, some and no, but no primitive name for not all. It is true also in the modal version of the square, we have necessary, possible and impossible, but no primitive name for not necessary.*

I shed here a new light on this mysterious non-lexicalisation of the south-east corner of the square of oppositions, the so-called O-corner, by establishing some connections between negations and modalities. The E-corner, *impossible*, is a paracomplete negation (*intuitionistic negation* if the underlying modal logic is S4) and the O-corner, *not necessary*, is a *paraconsistent negation*.

I argue that the three notions of opposition of the square of oppositions (contradiction, contrariety, subcontrariety) correspond to three notions of negation (classical, paracomplete, paraconsistent).

We get a better understanding of these relations of opposition in the perspective of Blanché's hexagon and furthermore in the perspective of a three dimensional object, a stellar dodecahedron of oppositions that I present at the end of the paper.

1. The non lexicalization of the O-corner of the Square of Oppositions

The famous Square of Oppositions¹ of Boethius and Apuleius with the four corners A E I O is built using the three Aristotelian notions of opposition: contradiction, contrariety and subcontrariety.

Contradiction is expressed in red, contrariety in blue, subcontrariety in green and subalternation² in black.

¹ Most of the time people use the singular, i.e. the expression "Square of Opposition". But I think it is important to emphasize the plurality of oppositions, that is why I will use the plural, as in French: "Le Carré des Oppositions".

² I represent subalternation, but I don't consider this relation as a relation of opposition, this topic will be discussed in section 3.

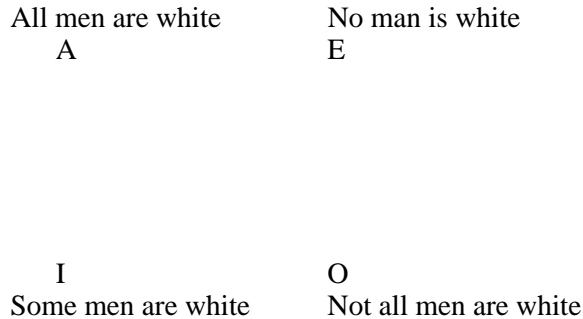


Figure 1. The traditional square of opposition

It has been pointed out that there is no primitive name in natural and formal languages for the O-corner of this square. We have *all*, *some* and *no*, but no primitive name for *not all*:

... striking ... is the observation in [Horn 1989, p.259] that natural languages systematically refuse to lexicalize the O-quantifier, here identified with “not all”. There are no known cases of natural languages with determiners like “nall”; meaning “not all”. Even in cases that look very promising (like Old English, which has an item *nalles*, derived from *alles*, “all”; by adding the negative prefix *ne-* the same that is used in words like *never*, *naught*, *nor*, *neither*), we end up empty-handed. *Nalles* does not actually mean “not all” or “not everything”, but “not at all” [Horn 1989, p.261]. Jespersen [1917] suggested that natural language quantifiers form a Triangle, rather than a Square (Hoeksema 1999, p.2).

In fact it seems reasonable to sustain that on the one hand there are only three main quantifiers in natural languages and that on the other hand the quantifier *some* of natural language is not correctly represented by the I-corner, because *some* implies *not all*: if someone says “Some cats are black” he doesn’t want to say “All cats are black”. This point was already made by Hamilton and Venn in the XIXth century and has been more systematically developed by Robert Blanché (1953, 57, 66). According to him, *some* has to be considered as the conjunction Y of the I and the O corners. The A E Y (*all*, *no*, *some*) vertices form a Triangle of Contrariety. But Blanché is not, like Vasiliev (1910) or Jespersen (1917), arguing for this Triangle against the traditional Square, he is instead supporting the idea of a Hexagon made of two triangles, the Triangle of Contrariety just described and a Triangle of Subcontrariety of which the O-corner is a vertex:

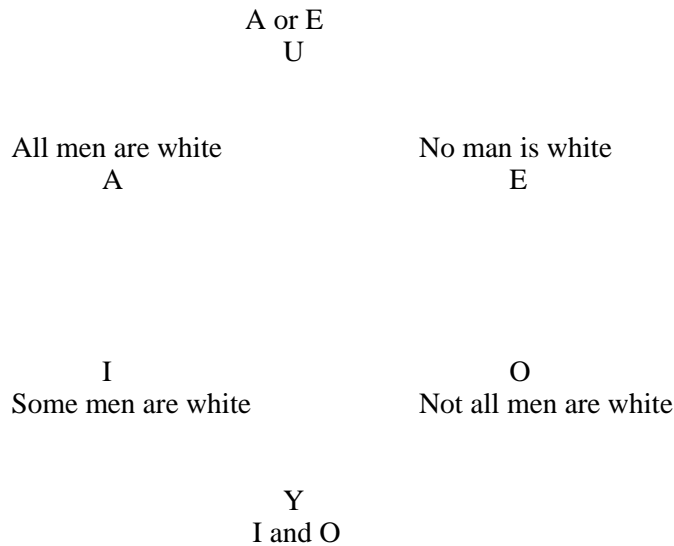


Figure 2. Blanche's hexagon

In the modal version of the Square of Oppositions, the O-corner is also not lexicalized: we have *necessary*, *possible* and *impossible*, but no primitive name for *not necessary*. Some similar remarks made about the quantificational version applied here. It seems that in natural language *possible* corresponds rather to the conjunction of the I and the O corners: when someone says “It is possible that it will rain”, he also means that “It is possible that it will not rain” i.e. that “It is not necessary that it will rain”. However nowadays people call *contingent* by contrast with *possible* this conjunction Y of I and O. But during several centuries people didn't make a distinction between possible and contingent and were rather considering a Triangle of Modalities corresponding in fact to the Triangle of Contrariety of Quantifiers (see Gardies 1979)³. A typical example, at the dawn of modern logic, is Wittgenstein in the *Tractatus* (4.464): he calls a proposition (*Satz*) something that can be true and can be false by contrast with a tautology and a contradiction and he says that for this reason the truth of the proposition is possible⁴.

³ We find a Triangle rather than a Square also in the case of temporal modalities (always, sometimes, never) and spatial modalities (everywhere, somewhere, nowhere). In the case of deontic modalities, this is not so clear, see e.g. (Chisholm 1963).

⁴ We see here a correspondence between modalities and quantifiers, since for Wittgenstein possible means for *some* bivaluations we have truth *but not for all*. An

We can also construct a Hexagon of Modalities applying here the ideas of Blanché:

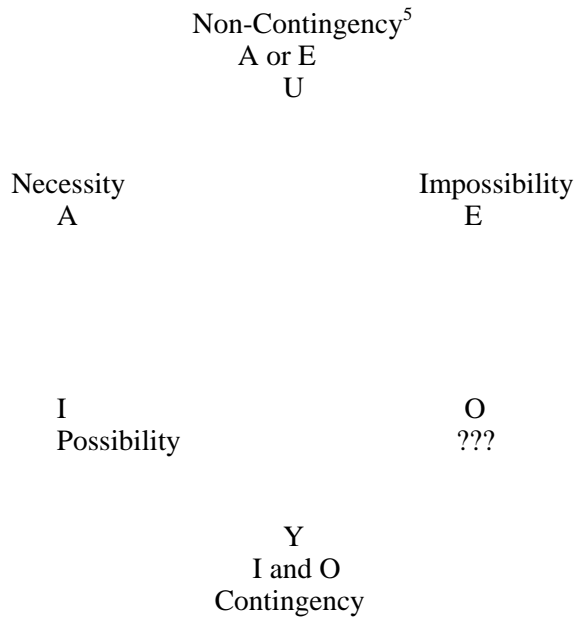


Figure 3. The hexagon of modalities

Now in the context of the modal version of the Square of Oppositions, we have some other interpretations of the corners: the E-corner, *impossible*, can be considered as a *negation*, it is in fact *intuitionistic negation* if the underlying modal logic is S4, as shown by Gödel (1933). I have shown recently that the O-corner, *not necessary* (or *possible not*) is a *paraconsistent negation*, for example in S5 (Béziau 2002). The modal logic S5 can be in fact reconstructed taking this paraconsistent negation as the only primitive modality⁶.

The fact that the corner corresponding to paraconsistent negation is not lexicalized is interesting. In some sense this could be used against the notion of paraconsistent negation, saying that it is not natural at all.

exact correspondence between the Square of Oppositions and the Square of Modalities is established by Wajsberg's theorem (1933).

⁵ There is also no primitive name for the contradictory of contingency, which is usually called "non-contingency".

⁶ The axiomatization of modal logics in terms of contingency and non-contingency was first examined by Montgomery and Routley (1966). For later works on this line, see for example (Cresswell 1988) and (Humberstone 1995).

But what is also interesting is that, even without name, this notion appears explicitly in the Square of Oppositions. What I want to show here is that paraconsistent negation seems quite natural in the context of the Square and vice-versa: the Square seems more natural if we observe that the O-corner can be interpreted as a paraconsistent negation. From this point of view, I will not argue for a Triangle instead of a Square. I will consider Triangles only in the broader context of a Hexagon. This paraconsistent view of the O-corner fits well in Blanché's Hexagon and we can even construct a more sophisticated geometrical object which explains quite well the relations between modalities and negations. This will be described in the last section.

2. Three oppositions and three negations

According to Aristotle's definitions two propositions P and Q are:

- *contradictories* iff they cannot both be true and cannot both be false.
- *contraries* iff they can both be false, but cannot both be true.
- *subcontraries* iff they can both be true, but cannot both be false.

Let us introduce some related definitions. A logical operator $\#$ is a

- *contradictory forming operator*, or *contradictory operator* for short, iff for every proposition P , P and $\#P$ are contradictories,
- *contrary operator* iff there is a proposition P , such that P and $\#P$ are false and for every proposition Q , Q and $\#Q$ cannot both be true.
- *subcontrary operator* iff there is a proposition P , such that P and $\#P$ are true and for every proposition Q , Q and $\#Q$ cannot both be false.

The Peruvian philosopher Francisco Miró Quesada has introduced the terminology "paraconsistent negation" and also the terminology "paracomplete negation"⁷. According to him, a negation \sim is *paraconsistent* iff there is a proposition P such that P and $\sim P$ can both be true and a negation \sim is *paracomplete* iff there is a proposition P such that P and $\sim P$ can both be false.

We can establish a clear one-to-one correspondence between logical operators defined from the three Aristotelian notions of opposition and

⁷ This terminology was introduced by Quesada in 1975 in a letter to Newton da Costa and became known through the work of the latter, see e.g. (Loparic and da Costa 1984) and (Grana 1990).

three kinds of negation: classical negation corresponding to the notion of contradictory operator, paracomplete negation to contrary operator and paraconsistent negation to subcontrary operator.

This correspondence is not perfect since a paraconsistent negation, according to Quesada's definitions, can also be a paracomplete one, but following our above definitions a subcontrary operator cannot be at the same time a contrary one. In fact Quesada also introduced the notion of *non-alethic negation*, it is a negation that is both paraconsistent and paracomplete. Let us say that a negation is a *proper paraconsistent* negation iff it is paraconsistent but not non-alethic and we introduce in a similar way the notion of *proper paracomplete* negation. This time we have a nice one-to-one correspondence between three kinds of negations and the three Aristotelian notions of oppositions via the three related notions of logical operators.

One may be against this kind of correspondence claiming that there is only one kind of negation, classical negation, and that proper paraconsistent and paracomplete negations are not negations. In fact Hartley Slater (1995) claimed that paraconsistent negations are not negations *because* they are only subcontrary operators. I think that, funnily enough, we can also claim exactly the contrary: paraconsistent negations are negations *because* they are subcontrary operators. This last claim is based on the idea that it is difficult to dissociate negation from opposition, that the background of negation is opposition and therefore if there are three kinds of oppositions there must also be three kinds of negations.

I don't think that we can really claim that paraconsistent and paracomplete negations (even non-alethic ones) are *in general* too weak to be called negations. Maybe it will be better to say that *some of them* are too weak. But in this case we lose the interesting connection between the Aristotelian notions of oppositions and negation. This connection is not only interesting from an historical point of view but also because this is a way of founding a theory of negation which allows paraconsistent and paracomplete negations to be considered as negations. At the present time we don't have any basis for such a pluralist theory of negation, there is no general definition of negation which includes classical, paracomplete and paraconsistent negations.

It seems to me that the connection between Aristotelian notions of oppositions and negation should not be broken anyway. If we claim that only some of the proper paraconsistent negations and proper paracomplete negations are negations, we should also consider that only some of the contrary operators and subcontrary operators are operators of oppositions, in other words, that contrariety and subcontrariety do not always express oppositions.

This is sound and not necessarily against Aristotle who was considering only some specific relations of contrariety and subcontrariety, since his theory was based only on some special propositions (universal affirmatives, universal negatives, particular affirmatives and particular negatives). Nevertheless Aristotle's theory itself presents several drawbacks.

3. Some controversies about the theory of oppositions

Aristotle's theory of oppositions is not very clear. Let us recall that Aristotle does not introduce explicitly the notion of "subcontraries", but refers to them only indirectly as "contradictories of contraries"; moreover he does not really consider them as opposed:

Verbally four kinds of opposition are possible, viz. universal affirmative to universal negative, universal affirmative to particular negative, particular affirmative to universal negative and particular affirmative to particular negative: but really there are only three: for the particular affirmative is only verbally opposed to the particular negative. Of the genuine opposites I call those which are universal *contraries*, e.g. 'every science is good', 'no science is good'; the others I call *contradictories*. (Aristotle, *Prior Analytics*, 63b21-30)

One may think that the reason why Aristotle considers "contraries" as opposites and not "subcontraries" is related to the fact that "contraries" are incompatible, they respect the principle of contradiction, although they do not respect the principle of excluded middle. It seems that Aristotle defends an asymmetrical view, privileging the principle of contradiction over the principle of excluded middle.

However from the point of view of modern formal logic, everything is symmetrical, or better, dual, in such a way that it makes no sense to say that contraries are opposed and subcontraries are not.

Aristotle's theory bears also some small incoherencies from the modern viewpoint. For example, as pointed out by Sanford (1968), a universal affirmative A is not necessarily contrary of a universal negative E: if we consider a universal affirmative which is a logical truth, it can never be false, so there are no propositions which are contraries of A⁸.

⁸ There are other problems with the Square of Oppositions, such that as what happens if the extension of the subject is void, etc. See e.g. (Parsons, 1999).

Also Aristotle mixes these two notions of oppositions (contradiction and contrariety) based on truth and falsity with some other kinds of oppositions. For example in the *Categories* (11b17), he considers four species of oppositions:

- correlation, e.g. *double* vs. *half*
- contrariety, e.g. *good* vs. *bad*
- privation, e.g. *blind* vs. *sighted*
- contradiction, e.g. *He sits* vs. *He does not sit*

Let us try to limit ourselves to a theory of oppositions based on truth and falsity. Can we say that another relation which appears through the doctrine of the Square (but not in Aristotle), subalternation, is a relation of opposition? Let us recall that a proposition P is *subaltern* to a proposition Q iff Q implies P. Q is sometimes called *superaltern* of P. I think that neither subalternation nor superalternation can be considered as relations of opposition. For example P is subaltern of $P \wedge Q$, and it does not really make sense to consider them as opposed.

Note that according to the definition of subcontrariety, P and $P \wedge Q$ are subcontraries, and also two tautologies are subcontraries. Similar problems happen with contrariety. So I think we have to improve the definitions by excluding subalterns and superalterns from subcontraries and contraries.

If we do not consider subalternation and superalternation as oppositions, Blanché's Hexagon seems a better representation of the relations of oppositions than the traditional Square, at least if we see it rather as a Star made of one Triangle of Contrariety and one Triangle of Subcontrariety. The sides of Blanché's Hexagon are relations of subalternation or superalternation, but that is not what is important, we can erase these sides and just stay with the Star, which represents only oppositions.

From this point of view I don't agree with the revised theory of oppositions presented by Avi Sion. He says that:

By the 'opposition' of two propositions, is meant: the exact logical relation existing between them – whether the truth or falsehood of either affects, or not, the truth or falsehood of the other.

In this context, note, the expression 'opposition' is a technical term not necessarily connoting conflict. We commonly say of two statements that they are 'opposite', in the sense of incompatible. But there, the meaning is wider; it refers to any mental confrontation, any logical face-off, between distinguishable propositions. In this sense, even forms which imply each other may be viewed

as ‘opposed’ by virtue of their contradistinction, though to a much lesser degree than contradictories. Thus, the various relations of opposition make up a continuum (Sion 1996).

According to Sion, there are six relations of opposition: contradiction, contrariety, subcontrariety, subalternation, impliance and unconnectedness. Here is his definition of this last concept:

Unconnectedness (or neutrality): two propositions are ‘opposed’ in this way, if neither formally implies the other, and they are not incompatible, and they are not exhaustive (Sion 1996).

In fact, according to this definition, two atomic propositions are unconnected, and must be considered as opposites, like “Snow is white” and “The sky is blue”, or “John likes cheese” and “John likes wine”. Obviously we are going too far and confusing here negation with distinction, maybe coming back to Plato’s theory in the *Sophist* where negation is identified with otherness.

The standard definition of *opposite* runs as follows:

A person or thing that is *as different as possible* from someone or something else: The colors ‘black’ and ‘white’ are opposites” (Longman dictionary of contemporary English. Italics are mine. – J.-Y.B.).

According to this definition, opposition is based on difference, but on *strong* difference. As we have here a matter of degree, we may be led to a kind of sorites paradox. So maybe it will be better to avoid such kind of degrees of difference.

Let us restrict ourselves anyway to the three main notions of oppositions: contradiction, contrariety and subcontrariety. Already here we have some problems, since some of these oppositions may appear too weak, even excluding subalternation and superalternation.

For example, in classical logic, P and $P \vee Q$ are subcontraries: obviously they cannot be false together, but they can be true together, when P is true and Q is true. So “God exists” and “If God exists, Satan exists” are subcontraries. Another example of subcontraries are P and $\neg P \vee \neg Q$: “God exists” and “God does not exist or Satan does not exist”.

Examples of contraries are P and $\neg P \wedge Q$: “God exists” and “God does not exist and Satan exists”; or P and $\neg P \wedge \neg Q$: “God exists” and “God does not exist and Satan does not exist”.

I think that we can say that contrariety and subcontrariety express sometimes some notions of opposition which are quite weak but it seems that we may still argue that they express a kind of opposition and

we can also argue that there are correlated paraconsistent and paracomplete negations corresponding to them. In particular there are such negations within classical logic, as shown by the above examples⁹.

4. The O-corner within the Octagonal and the Stellar Dodecahedron of Oppositions

The notions of opposition presented in the traditional Square of Oppositions are oppositions between quantified propositions. In the Square of Modalities they express oppositions between modal propositions. The oppositions between a proposition and its negations (whether classical, paraconsistent and paracomplete) do not appear in the Square. In the Square of Modalities, if we see the E-corner as a paracomplete negation, intuitionistic negation in the case of S4, this negation (i.e. the negation $\sim P$ of a proposition P) appears as contrary to necessity (i.e. the necessitation $\Box P$ of the proposition P). But what we want to express is the opposition between $\sim P$ and P . However P does not appear in the Square of Oppositions. One possibility would be to add P and $\neg P$, the classical negation, to the Square in the following way:

Figure 4.

See below.

However this figure does not look very nice. To get a better result, we consider three Hexagons constructed with the same idea underlying Blanché's Hexagon, i.e. with one Triangle of Contrariety and one Triangle of Subcontrariety. Since we are not directly interested in subalternation, we will rather consider them as Stars. The first Star is in fact the one that can be found in Blanché's Hexagon:

Figure 5.

See below.

⁹ The notions of opposition from the Square have been used to explain the relations between binary connectives of classical logic, see e.g. (Blanché 1957b).

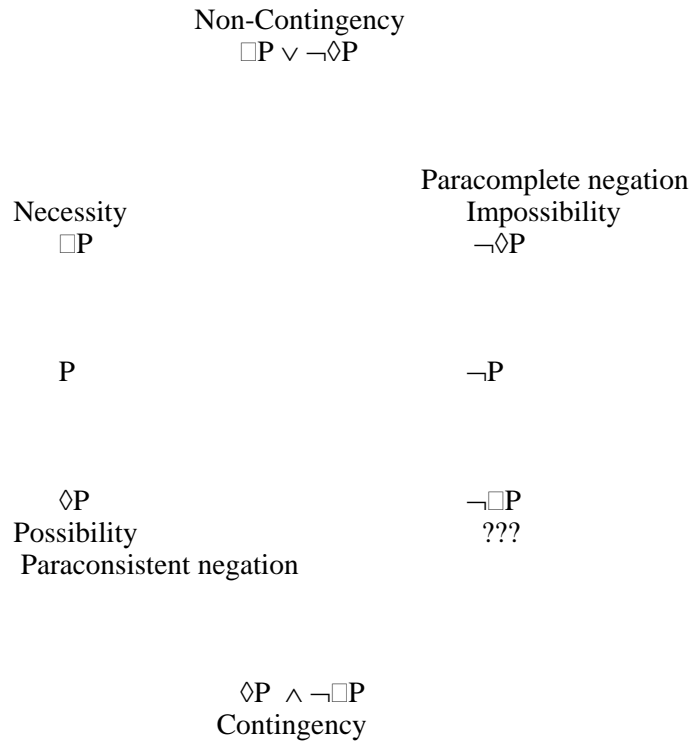


Figure 4. The octagon of modalities

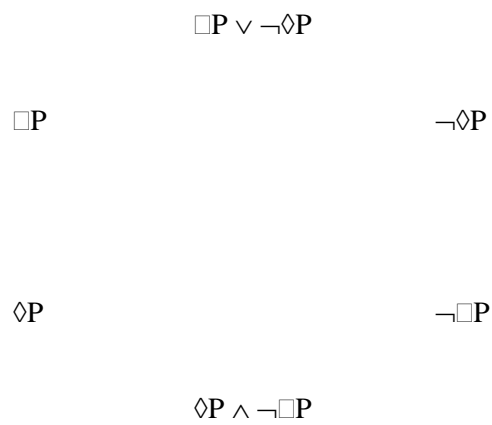


Figure 5. Blanche star of oppositions

The two following Stars are, on the one hand the Star establishing the connections between P , the classical negation $\neg P$ and the paracomplete negation $\neg\Diamond P$, and on the other hand the Star establishing the connections between P , the classical negation $\neg P$, and the paraconsistent negation $\neg\Box P$:

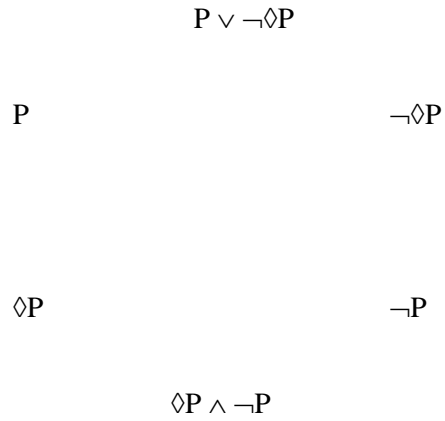


Figure 6. Paracomplete star of oppositions

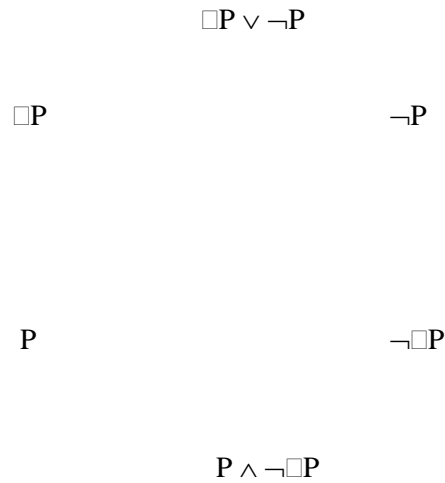


Figure 7. Paraconsistent star of oppositions

Among the 18 vertices of these three Stars, 6 appear two times. So if we link these three Stars together in a three-dimensional way by putting

together the vertices appearing two times we get an object with 12 vertices. The three Stars are tied together by 6 relations of contradiction. The corresponding polyhedron is a Stellar Dodecahedron (the first stellation of the rhombic dodecahedron precisely).

This Stellar Dodecahedron of Oppositions permits to have a better understanding of the concept of negation in its plurality and of its relation with possibility and necessity, by presenting the full oppositions between 12 basic unary connectives.

5. Conclusions

Let us summarize the main conclusions of this investigation:

- i) The nameless corner of the Square of Oppositions is a paraconsistent negation.
- ii) The oppositions either at the level of quantifiers or modalities are better represented by Blanché's Hexagon of Oppositions.
- iii) The full relations of oppositions between modalities and negations are better represented by a Stellar Dodecahedron of Oppositions.
- iv) The three notions of oppositions which appear in the Square, the Hexagon and the Dodecahedron, namely contradiction, contrariety and subcontrariety, correspond to three notions of negations, respectively, classical negation, paracomplete negation and paraconsistent negation.
- v) It makes no sense to argue that contrariety is a relation of opposition, but that subcontrariety is not.
- vi) Subalternation and supalternation are not relations of opposition.

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¹⁰ His booklet has been (partially) translated in Portuguese in (Arruda 1990) and is discussed in (Lucchese and Grana 2000).